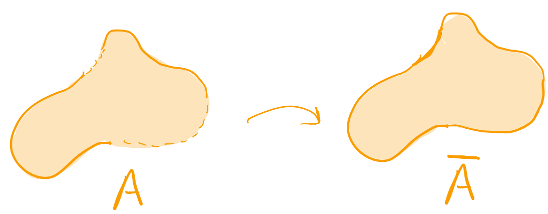
Closed sets

X a topological space. $A \subseteq X$ is <u>closed</u> if $X \setminus A$ is open.
Note that sets can be simultaneously open and closed (e.g. ϕ and X).

Def: $A \subseteq X$ any subset.

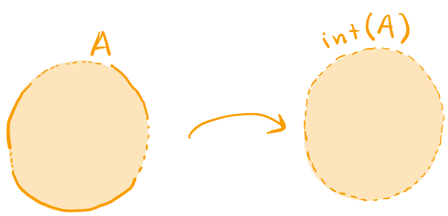
1.) The <u>closure</u> of A, denoted $\overline{A}$ is the smallest closed set containing A.
i.e. $\overline{A} = \underbrace{\bigcap_{\substack{A \subseteq V \\ V \text{ closed}}} V}$

closed – intersection of closed sets

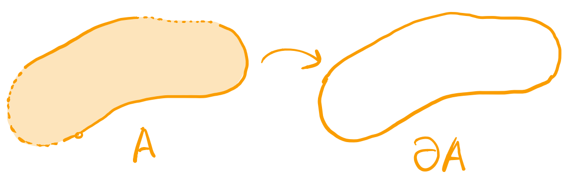


2.) The <u>interior</u> of A, denoted $\text{int}(A)$, is the largest open set contained in A. i.e. $\text{int}(A) = \underbrace{\bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U}$

union of open sets $\Rightarrow$ open



3.) The <u>boundary</u> of A (denote ∂A or $Bd(A)$) is $\overline{A} - \text{int}(A)$



4.) $A \subseteq X$ is <u>dense</u> in X if $\overline{A} = X$.

Ex: $A = [0,1) \subseteq \mathbb{R}$. Then
$\overline{A} = [0,1]$, $\text{int}(A) = (0,1)$, and $\partial A = \{0,1\}$.

Ex: $\mathbb{Q} \subseteq \mathbb{R}$ given the subspace topology.

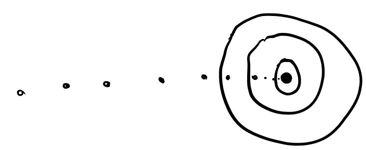
Consider $\overline{\mathbb{Q}} \subseteq \mathbb{R}$. suppose $x \in \mathbb{R}\setminus\overline{\mathbb{Q}}$, which is open.
Thus, $\exists$ $x \in (a,b) \subseteq \mathbb{R}\setminus\overline{\mathbb{Q}}$, but there's some rational number in (a,b).

Thus, $\mathbb{R}\setminus\overline{\mathbb{Q}} = \emptyset \Rightarrow \overline{\mathbb{Q}} = \mathbb{R} \Rightarrow \overline{\mathbb{Q}}$ is dense in $\mathbb{R}$.

Rmk: A is closed $\iff \overline{A} = A$.

Limit Points

In a metric space, we have the notion of the limit of a sequence



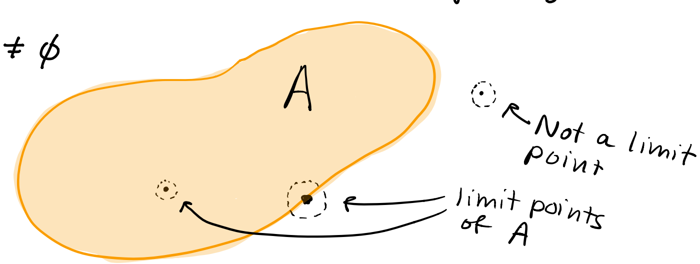
$x_n \to x$ if $\forall\, \varepsilon > 0\ \exists\ N$ s.t. for $n \geq N$,
$|x_i - x| < \varepsilon$

We also proved there is an equivalent definition involving open sets: $x_n \to x$ if every open $U \ni p$ contains all but finitely many x_i.

We want a notion of limit points of sets more generally.

Def: 1.) X a topological space, $x \in X$. U is a <u>neighborhood</u> of x if $x \in U \subseteq X$ and U is open

2.) If $A \subseteq X$, $x \in X$ is a limit point of A if for every neighborhood U of x, $U \cap (A \setminus \{x\}) \neq \emptyset$

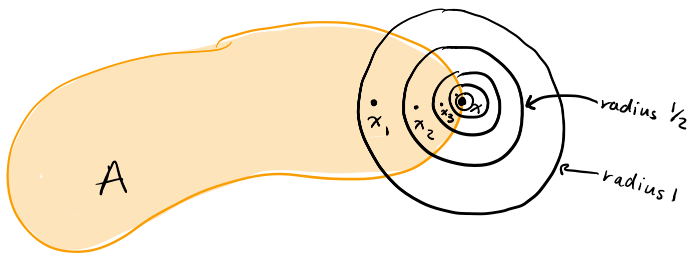


Ex: 1 is a limit point of $[0,1)$ and of $[0,1]$.

1 is not a limit point of $A = (0, \tfrac{1}{2}) \cup \{1\}$ since $(\tfrac{3}{4}, \tfrac{5}{4}) \cap A = \{1\}$.

Remark: If X is a metric space, and $A \subseteq X$. If x is a limit point of A, $\forall\, n$ we can find $x_n \neq x$ s.t. $x_n \in B_{1/n}(x) \cap A$.

Thus $x_1, x_2, \ldots$ converges to x, so it's also the limit of a sequence in A.



Thm: $\bar{A} = A \cup \{\text{limit points of } A\}$.

Pf: Suppose $x \notin A$ and x not a limit point.

Then $\exists$ a neighborhood U of x s.t. $U \cap A = \emptyset$. Thus

$$A \subseteq \underbrace{X \setminus U}_{\text{closed}} \Rightarrow \bar{A} \subseteq X \setminus U, \quad \text{so} \quad U \cap \bar{A} = \emptyset \quad \text{so} \quad x \notin \bar{A}.$$

For the other inclusion, assume $x \notin \bar{A}$. Then $U = X \setminus \bar{A}$ is an open neighborhood of x disjoint from A, so x is not a limit point. $\square$

Limits of sequences

X an arbitrary topological space. Then $x_1, x_2, \ldots$ <u>converges</u> to x if for every neighborhood U of x

$\exists \; N_U$ s.t. $\forall \; n \geq N_U \quad x_n \in U$.

For an arbitrary topological space, not all limit points of A are limits of sequences in A!

Ex: Let $X = \mathbb{R}$ with topology

$$\mathcal{T} = \{ U \mid X \setminus U \text{ is } X \text{ or countable} \}$$

(Check that this satisfies the axioms)

Let $B = (0,1)$. Then $\overline{B} = X$, so 2 is a limit point!

But there is no sequence $a_n \in B$ s.t. $a_n \to 2$, since the complement of that sequence is an open neighborhood of 2.

Hausdorff Spaces

Recall: In a metric space, every sequence can converge to at most one point.

However, this is not true in an arbitrary topological space.

Ex: Consider $X = \mathbb{R}$ w/ the finite complement topology.

Let $a_1, a_2, \dots$ be a sequence.

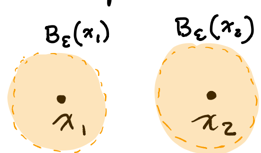
Then $\forall \; x \in X$, a neighborhood U of x contains all but finitely many a_i, so there is some N s.t. $a_n \in U \quad \forall \; n \geq N$.

Thus the sequence converges to <u>every</u> point of X.

We can avoid this by requiring our space to be Hausdorff:

Def: A top. space X is <u>Hausdorff</u> if any pair of distinct points x_1 and x_2 have disjoint neighborhoods U_1 and U_2, respectively.

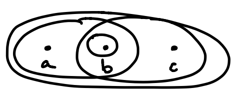
Ex: 1.) Any metric space is Hausdorff



Choose $0 < \varepsilon < \frac{1}{2} d(x_1, x_2)$

2.) The cofinite topology on $\mathbb{R}$ is not Hausdorff since every pair of nonempty open sets intersect (in infinitely many points).

3.) X w/ discrete topology is always Hausdorff. $(U_1 = \{x_1\},\ U_2 = \{x_2\})$

4.) 

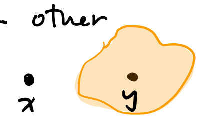
← This 3-point set is not Hausdorff.

Thm: If X is Hausdorff, every sequence converges to at most one point.

Pf: If $x_1, x_2, \ldots$ converges to x and $x \neq y$, choose $U_x \ni x, U_y \ni y$ disjoint neighborhoods. Then there's some N s.t. $x_n \in U_x$ $\forall\, n \geq N$. Thus $x_N, x_{N+1}, \ldots \notin U_y$, so the sequence doesn't converge to y. $\square$

Historical note: Before current def. of topological space, there were several other proposed definitions, some of which assumed various axioms, now called the separation axioms:

- **X is T_0** if $\forall$ x, y distinct, one of them has a neighborhood not containing the other



- **X is T_1** if all one point sets are closed

- **X is T_2** if X is Hausdorff.

Clearly $T_1 \Rightarrow T_0$ since $T_1 \Longrightarrow X \setminus \{x\}$ is a neighborhood of y.

Thm: If X is Hausdorff, then every one-point set is closed.

i.e. $T_2 \Rightarrow T_1$.

Pf: Let $x \in X$. Then $\forall$ $y \in X$ s.t. $x \neq y$, can find a neighborhood U that doesn't contain x.

$\Rightarrow$ $U \cap \{x\} = \phi$, so y is not a limit point of $\{x\}$, so $\overline{\{x\}} = \{x\}$

$\Rightarrow$ $\{x\}$ is closed. $\square$

The converse doesn't hold:

Ex: Cofinite topology is T_1 but not Hausdorff.